Coalition Formation in Proportionally Fair Divisible Auctions *

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ABSTRACT

One method for agents to improve their performance is to form coalitions with other agents. One reason why this might occur is because different agents could have been created by the same owner so an incentive to cooperate naturally exists. Competing agents can also choose to coordinate their actions when there is a mutually beneficial result. The emergence and effects of cooperation depend on the structure of the game being played. In this paper, we study a proportionally fair divisible auction to manage agents bidding for service from network and computational resources. We first show that cooperation is a dominant strategy against any fixed level of competition. We then investigate whether collusion can undermine a noncooperative equilibrium solution, i.e. allow an agent priced out of the noncooperative game to enter the game by teaming with other agents. We are able to show that agents not receiving service after a bid equilibrium is reached cannot obtain service by forming coalitions. However, cooperation does allow the possibility that agents with positive allocations can improve their performance. To know whether or not to cooperate with another agent, one must devise a way of assigning a value to every coalition. In classical cooperative game theory, the value of a team is the total utility of the team under the worst case response of all other agents, as a coalition is viewed as a threat by the remaining agents. We show that this analysis is not appropriate in our case. The formation of a coalition under a proportionally fair divisible auction improves the performance of those outside the coalition. This then creates an incentive structure where team play is encouraged.

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1. INTRODUCTION

An emerging application for software agents is serving as proxies for trade [8]. Mechanisms to serve this end [1] and analysis of automated trading [10] has been an area of active research. Agents are not limited to procurement of traditional goods such as flights, books and anything sold on E-bay. Markets have been advocated as a tool for controlling electronic resources [2]. We focus on computational and network resources. In the computational regime, markets have been proposed as a scheme for decentralized regulation of CPU time [3,18]. In the realm of networks, there has been a push to use pricing and economic modeling to address congestion issues of the Internet [4, 11].

The requirements for partitioning electronic resources are not satisfied by traditional mechanisms as the goods being sold (such as processing time and bandwidth) are essentially divisible. Single-good and combinatorial auctions do not readily apply as these resources are usually assigned to multiple agents in bundle sizes that cannot be known *a priori*. We have investigated a proportionally fair auction that yields a unique Nash equilibrium and developed decentralized schemes that converge to the operating point [13–15].

The assumption in the analysis was that agents acted in a purely noncooperative manner. However, it is possible for agents to improve their performance by acting in teams. When implementing a mechanism, one must fully understand the consequences of cooperative or collusive behavior. Team analysis can also give bounds on performance of agents in noncooperative games. Thus, coalition formation has been a rich area for investigation [5, 9].

Cooperative phenomena depend on the structure of the game being played. Our goal is to investigate the effects of coalition formation in the regime of the proportionally fair divisible auction to answer the following questions. How does one obtain the value of a coalition? What is the reaction of other agents to coalitions? How do coalitions affect the purely noncooperative equilibrium?

In classical cooperative game theory [16, 17], the value of a team is the total utility of all agents in the team, under the worst case response of all other agents. This value is assigned due to an assumption that agents outside the team will see the formation of a coalition as a threat and will do everything in their power to discourage cooperation. We are able to show that this assumption is not appropriate in the proportionally fair divisible auction.

The paper is organized as follows. In Section 2, we introduce the model and present some background results. In Section 3, we show that forming a coalition is a dominant strategy given any fixed strategy of other agents. We also show that a team of agents can be modeled as a single agent with a modified valuation function. In Section 4, we address the effects of coalition formation. We show that joining a coalition does not increase an agent's ability to obtain service, if it was excluded in a purely noncooperative game. We also prove that forming a team improves the performance of those outside the team. This contradicts the punitive approach to coalition valuation. In Section 5, we present some concluding thoughts and open questions.

2. MODEL AND BACKGROUND

In this section, we introduce the model, characterize an agent's optimal response and show the existence of a unique Nash equilibrium. We begin with N agents competing for a resource with fixed finite capacity. The resource is allocated using a market mechanism, where the partitions depend on the relative signals or bids sent by the agents. We assume that each agent submits a nonnegative real bid s_i to the resource. A divisible auction consists of two mappings. The first is from the bids, s, to a partition, $x(s)$, where $x_i(s) \in [0, 1]$ is the resource share allocated to the i-th bidding agent. The second is from the bids, s, to a cost vector, $c(s)$ where $c_i(s)$ is the cost associated with the i-th agent. The choices of x and c define the auction mechanism.

In designing our auction, we want our allocations to be *proportionally fair* by weight. Proportional fair allocations in networks was introduced by Kelly [6, 7] and has generated support as a congestion control criterion for the internet. Furthermore, proportionally fair allocations are advocated as an efficient method for partitioning computational resources [12, 20].

This can be achieved with the following allocation rule:

$$x_i(s) = \frac{s_i}{\sum_j s_j}. \tag{1}$$

The cost for each agent is the bid

$$c_i(s) = s_i. \tag{2}$$

If the feedback from the resource is the sum of all bids, an agent can immediately verify if it has been given an accurate allocation. If an agent knows the received allocation x_i and its own bid s_i, any bid total suggested by the auctioneer other than s_i/x_i can be immediately identified as a signal of an inaccurate allocation or a lying auctioneer [19]. Furthermore, under this cost structure, each agent pays the same price per unit resource received.

We assume that each agent has a valuation $v_i(x_i)$ for receiving an allocation x_i. This valuation may be a characterization of the estimated performance as a function of a given share of the resource. For example, it could be the time to complete the processing of a job in a computational market, or the time to transmit some data given a particular share of network bandwidth obtained. Each performance measure is translated to an equivalent value that can be compared with cost. Another derivation of the valuation could come from the estimated value of the sales that could be generated by obtaining a given share of the resource. This could be the case where agents act as brokers for computational resources or network bandwidth. We make the following assumptions about agents' valuations:

Assumption 1. *For all $i \in \{1, \cdots, N\}$*

- *$v_i(x_i)$ is continuously differentiable*
- $v_i'(x_i) > 0 \quad \forall\ x_i \in (0, 1)$
- $v_i''(x_i) \leq 0 \quad \forall\ x_i \in (0, 1)$

The first assumption captures the fact that an agent's performance or marginal valuation of performance should not change dramatically given a miniscule change in allocation. The second assumption is intuitive as an agent's valuation should increase with allocation, and the third assumption captures the effect of diminishing returns. Each agent's utility is the difference between the valuation and cost of its allocation:

$$U_i(s) = v_i(x_i(s)) - c_i(s, x).$$

Substituting from Equations (1) and (2), we have:

$$U_i(s) = U_i(s_i; s_{-i}) = v_i\left(\frac{s_i}{s_i + s_{-i} + \epsilon}\right) - s_i \tag{3}$$

where $s_{-i} = \sum_{j=1}^{N-1} s_j - s_i$ is the sum of the bids of all agents excluding the i-th agent and $s_N = \epsilon$ is a bid made by an agent representing the resource.

By bidding ϵ, the seller maintains a portion of the resource for its own use in addition to setting a minimum level of generated revenue. Specifically, a seller will receive at least $x\epsilon/(1-x)$ for allocating a total of x to agents 1 through $N-1$. This is analogous to declaring a reservation value in single-unit auctions.

The first order necessary condition for a maximizing interior solution is:

$$\begin{aligned} U_i'(s_i; s_{-i}) &= v_i'(x_i(s))x_i'(s) - 1 \\ &= v_i'\left(\frac{s_i}{s_i + s_{-i} + \epsilon}\right)\frac{s_{-i} + \epsilon}{(s_i + s_{-i} + \epsilon)^2} - 1 = 0. \end{aligned}$$

This can be rewritten as follows:

$$v_i'\left(\frac{s_i}{s_i + s_{-i} + \epsilon}\right) - \frac{(s_i + s_{-i} + \epsilon)^2}{s_{-i} + \epsilon} = 0.$$

The LHS of the above equation is a decreasing function of s_i as $v_i'(\cdot)$ is decreasing in its argument, $s_i/(s_i + s_{-i} + \epsilon)$ is an increasing function of s_i and the second term has s_i only in the numerator. Furthermore, the LHS tends to $-\infty$ as $s_i \to \infty$. Thus, an interior solution exists if and only if the LHS is positive when $s_i = 0$. An agent will participate in the auction (submit a nonzero bid), if and only if:

$$v_i'(0) > s_{-i} + \epsilon. \tag{4}$$

If this condition is satisfied, there exists an interior extremal point. Looking at the second order condition, we have:

$$U_i''(s_i; s_{-i}) = v_i''\left(\frac{s_i}{s_i+s_{-i}+\epsilon}\right)\frac{(s_{-i}+\epsilon)^2}{(s_i+s_{-i}+\epsilon)^4} + v_i'\left(\frac{s_i}{s_i+s_{-i}+\epsilon}\right)\frac{-2(s_{-i}+\epsilon)}{(s_i+s_{-i}+\epsilon)^3} \quad (5)$$

which is negative due to Assumption 1, and the nonnegativity of the bids, and the assumption that $\epsilon > 0$. Then, the extremal point is the unique value maximizing the agent's utility and is the agent's unique response when the total of all other agents' bids is s_{-i} and the resource bids ϵ. In the remainder of this paper, for notational simplicity, we will assume that the resource's bid is captured in the term s_{-i}. In the previous analysis, we have characterized an agent's optimal response as a function of s_{-i}. The response curve ultimately represents cost-allocation pairs which are acceptable to the agent. We now investigate alternate methods of representing these optimal cost-allocation pairs.

Even though the resource allocation is accomplished via an auction mechanism, we note that ultimately each agent pays the same price per unit resource obtained. The auction can then be interpreted as a resource sold at a uniform price where the price is determined by the agents. The price per unit of the resource is $\sum_i s_i$, and each agent receives an allocation in proportion to that price. We can then define the *price* of a resource as the sum of all the bids, including the resource's bid, for that resource.

Next we define a *price function*, $p_i(x_i)$, as the price at which the agent would choose an allocation of x_i. The price function represents the set of cost-allocation pairs which are the unique optimal responses of a given agent over a range of bids of other agents, i.e., $s_i = p_i(x_i)x_i$ is the unique optimal response to $s_{-i} = p_i(x_i)(1 - x_i)$. The inverse of the price function is the *demand function*, $d_i(p)$, which is defined as the quantity of resource that the agent would desire if the price was p. This is again generated by an agent's unique optimal response in a way such that $s_i = d_i(p)p$ is the agent's reaction to $s_{-i} = (1 - d_i(p))p$. The price and demand functions are expected to be differentiable decreasing functions of their argument and the existence of one implies the existence of a well-defined inverse. One way to obtain these functions is to take the optimal response $s_i = f_i(s_{-i})$, substitute $s_{-i} = p - s_i$, and solve the fixed point equation $s_i = f_i(p - s_i)$. If a solution exists, one has s_i as a function of p. Then, making the substitution $s_i = px_i$, one can obtain an equation in terms of x_i and p from which the price and demand functions can be obtained. However, due to the nature of our auction, we can obtain the price function directly from an agent's valuation.

Proposition 1. *Given a valuation $v_i(x_i)$ that satisfies Assumption 1, there exists a corresponding differentiable decreasing price function characterized by $p_i(x_i) = v_i'(x_i)(1 - x_i)$.*

Proof. Let $f_i(s_i)$ be the i-th agent's unique optimal response. By the first order necessary condition, we have:

$$f_i(s_{-i}) + s_{-i} = v_i'\left(\frac{f_i(s_{-i})}{f_i(s_{-i}) + s_{-i}}\right)\frac{s_{-i}}{f_i(s_{-i}) + s_{-i}} = v_i'\left(\frac{f_i(s_{-i})}{f_i(s_{-i}) + s_{-i}}\right)\left(1 - \frac{f_i(s_{-i})}{f_i(s_{-i}) + s_{-i}}\right).$$

By the definition of price, $p_i = f_i(s_{-i}) + s_{-i}$, and the allocation rule states $x_i = f(s_{-i})/(f_i(s_{-i}) + s_{-i})$. Substituting this above, we have:

$$p_i(x_i) = v_i'(x_i)(1 - x_i).$$

We see that p_i is differentiable, with derivative:

$$p_i'(x_i) = v_i''(x_i)(1 - x_i) - v_i'(x_i)$$

which is strictly negative given Assumption 1. Thus, the price function is decreasing. ■

This property of the auction lets us go directly from knowing an agent's valuation to the price function which is a transformation of its optimal response. We can obtain the optimal bid from the price function as follows:

$$s_i = f_i(s_{-i}) = f_i(s_{-i})\frac{f_i(s_{-i}) + s_{-i}}{f_i(s_{-i}) + s_{-i}} = x_i p_i(x_i) = v_i'(x_i)(1 - x_i)x_i.$$

Given that we can obtain agents' optimal responses (and express them as bids which are functions of competing bids, allocation or price), an immediate question is whether there is an allocation of the resource at a cost where all agents participating in the auction are satisfied. In the language of game theory, we ask whether there is a set of bids $\{s_i^*\}_{i=1}^N$, where N is the number of agents competing for the desired resource, such that no single agent wishes to deviate from its bid given that the other agents remain the same. This state, a Nash equilibrium, occurs if no agent can improve its utility by changing its bid under current market conditions, i.e.,

$$s_i^* = \arg\max_{s_i} U_i(s_i; s_{-i}^*) \quad \forall\ i\ \in \{1, \cdots, N\}$$

where s_{-i}^* implies $s_j = s_j^*$, $\forall\ j \neq i$. Because every agent's optimal response is captured in its price and demand functions, we can use these as tools to evaluate the existence of a Nash equilibrium.

We find it useful to work in the space of demand functions. Due to the structure of our auction, the total of the allocated resources will always be one. If the optimal demand total at a particular price is greater or less than one, the allocation made to at least one agent will be unsatisfactory. Thus, it is equivalent to ask whether there is a price (or bid total) where the total demand of all the agents at that price is equal to one. Valid demand functions for elastic agents are assumed to be decreasing functions of price that go to zero as the price tends to infinity.

Proposition 2. *Given any set of demand functions $\{d_i(\theta)\}_{i=1}^N$, where $\sum_{i=1}^N d_i(0) > 1$, $lim_{\theta\to\infty} d_i(\theta) = 0, \forall\ i$, and $d_i(\theta_1) > d_i(\theta_2)\ \forall\theta_1, \theta_2$ such that $\theta_1 < \theta_2$ for $i = 1, \ldots, N$, there exists a unique value θ^* such that $\sum_{i=1}^N d_i(\theta^*) = 1$.*

Proof. Let $\bar{d}(\theta) = \sum_{i=1}^N d_i(\theta)$. Then $\bar{d}(\theta)$ is a continuously decreasing function whose maximum is $\bar{d}(0) > 1$. We also have $\lim_{\theta\to\infty} \bar{d}(\theta) = 0$ which implies that for some $\bar{\theta}$ sufficiently large, $\bar{d}(\bar{\theta}) < 1$. Applying the Intermediate Value Theorem for $\bar{d}(\theta)$ on $[0, \bar{\theta}]$, we know that there exists at least one θ^* such that $\bar{d}(\theta^*) = \sum_{i=1}^N d_i(\theta^*) = 1$. Let us assume that there are at least two values of θ where $\bar{d}(\theta) = 1$. Let us choose two of these values as θ_1^* and θ_2^*, where $\theta_1^* < \theta_2^*$. Then, we have $d_i(\theta_1^*) > d_i(\theta_2^*)\ \forall i = 1, \ldots, N$, which implies that $\bar{d}(\theta_1^*) > \bar{d}(\theta_2^*)$. But we have $\bar{d}(\theta_1^*) = \bar{d}(\theta_2^*) = 1$, which is a contradiction and thus we can have only one θ where $\bar{d}(\theta) = \sum_{i=1}^{K} d_i(\theta) = 1$. ■

By working in the space of demand functions, we can use the property that the demands are decreasing to easily see that there is a unique Nash equilibrium. Uniqueness of the Nash equilibrium is significant as we have a single desired operating point. Thus, given any set of agents there is a unique set of bids that yield an allocation where each agent is satisfied. This set of bids can be characterized in terms of the demand functions and Nash equilibrium price, θ^*, as follows:

$$\{s_i : s_i = d_i(\theta^*)\theta^*\}_{i=1}^{N}.$$

The condition $\sum_{i=1}^{N} d_i(0) > 1$ is satisfied for almost all agents as (for price function p), $p(1) = 0 \Rightarrow d(0) = 1$ unless the marginal valuation at one is infinite which will not occur for any reasonable valuation. This also requires that $N > 2$, and this is always satisfied as we have the bids of the resource and at least one agent requesting service. Clearly, θ^* determines which agents receive service as any agent with $d(\theta^*) = 0$ will have a zero bid as its optimal response. In [14], we investigate the optimal responses for agents with a series of jobs in an itinerary with varying objective functions and derive equivalent valuations. We also develop a locally stable decentralized update scheme that converges to the unique Nash equilibrium.

3. TEAMS OF AGENTS

In this section, we characterize team play by showing that acting as a team is a dominant strategy and a coalition can be modeled as a single agent. Given a set of N competing agents, let $\mathcal{N} = \{1, \cdots, N\}$, and $T \subset \mathcal{N}$ be a team or coalition of agents, where $N_T = |T|$ is the number of agents in the team. The i^{th} agent attempting to maximize its own utility would employ the strategy $s_i = f_i(s_{-i}) = \arg\min_{s_i} U_i(s_i; s_{-i})$, where $s_{-i} = \sum_{j \neq i} s_j$. Then, $s_T = \sum_{i \in T} s_i = \sum_{i \in T} f_i(s_{-i})$ would be the total bid of all the agents in the team and $s_{-T} = \sum_{i \in T^C} s_i$ would be the total of all the bids made by agents not in the team.

The demand function of the i^{th} agent $d_i(p)$ is the inverse of the price function $p_i(x_i) = v_i'(x_i)(1 - x_i)$, where $v_i(x_i)$ is the valuation of the i^{th} agent. The optimal individual bid can be represented as

$$f_i(s_{-i}) = d_i(s_T + s_{-T})(s_T + s_{-T}) \Rightarrow$$
$$\sum_{i \in T} d_i(s_T + s_{-T}) = \frac{s_T}{s_T + s_{-T}}$$

as we sum over all agents in the team. The LHS of the previous equation is a decreasing function of s_T and the RHS is an increasing function of s_T. Assuming $\sum_{i \in T} d_i(s_{-T}) > 0$, there is a unique solution for s_T given any value of s_{-T}. From this, we can obtain a function $f_T(s_{-T})$ that yields the total bid of the team when all the agents act with respect to their own utility functions given a particular level of competition from those outside the team. Let

$$X_T^f = \left\{ x : \sum_{i \in T} x_i = \frac{f(s_{-T})}{f(s_{-T}) + s_{-T}},\ x_i \geq 0\ \forall\ i \right\}$$

denote the set of all viable allocations when the team responds to s_{-T}, with the function $f(s_{-T})$. Then we have

$$\begin{aligned}
&\sum_{i \in T} U_i(f_i(s_{-i}); s_{-i}) \\
&= \sum_{i \in T} v_i\left(\frac{f_i(s_{-i})}{f_i(s_{-i}) + s_{-i}}\right) - \sum_{i \in T} f_i(s_{-i}) \\
&= \sum_{i \in T} v_i\left(\frac{f_i(s_{-i})}{f_i(s_{-i}) + s_{-i}}\right) - f_T(s_{-T}) \\
&\leq \max_{x \in X_T^{f_T}} \sum_{i \in T} v_i(x_i) - f_T(s_{-T}) \\
&\leq \max_{f \in F} \left\{ \max_{x \in X_T^f} \sum_{i \in T} v_i(x_i) - f(s_{-T}) \right\} \\
&= \max_{s_T} \left\{ \tilde{v}\left(\frac{s_T}{s_T + s_{-T}}\right) - s_T \right\}
\end{aligned}$$

where

$$\tilde{v}(\bar{x}) = \max_{x \in Z_T(\bar{x})} \sum_{i \in T} v_i(x_i) \quad (6)$$

for

$$Z_T(\bar{x}) = \left\{ x : \sum_{i \in T} x_i = \bar{x},\ x_i \geq 0\ \forall\ i \right\}.$$

Intuitively, the first inequality above states that the team would perform better if it reallocated the share obtained while bidding individually, in a manner that would maximize the team's objective. The second inequality states that the performance would increase further if the team bid according to the team's valuation, as opposed to bidding with respect to individual valuations. The valuation function for a team of agents is characterized by (6). This analysis shows that a team can be modeled as a single user with the valuation $\tilde{v}(\bar{x})$. To be complete, we must verify that this valuation function is concave, increasing, and differentiable.

LEMMA 1. *If $v_i(x_i)$ satisfies Assumption 1 $\forall i \in T$, then $\sum_{i \in T} v_i(x_i)$ is concave.*

Proof. Let x and y denote two allocation vectors and $\alpha \in (0, 1)$. Then,

$$\begin{aligned}
&\sum_{i \in T} v_i(\alpha x_i + (1 - \alpha)y_i) \\
&\geq \sum_{i \in T} \alpha v_i(x_i) + (1 - \alpha)v_i(y_i) \\
&\geq \alpha \sum_{i \in T} v_i(x_i) + (1 - \alpha) \sum_{i \in T} v_i(y_i) \quad ■
\end{aligned}$$

LEMMA 2. *Let $z \in (0, 1)$. Then,*

$$\tilde{v}(z) := \max_{\sum_{i \in T} x_i = z} \sum_{i \in T} v_i(x_i)$$

is concave.

Proof. Let $z_1, z_2, \alpha \in (0,1)$. Then,

$$\begin{aligned}
& \tilde{v}(\alpha z_1 + (1-\alpha) z_2) \\
= & \max_{\sum_{i\in T} x_i = \alpha z_1 + (1-\alpha) z_2} \sum_{i\in T} v_i(x_i) \\
= & \max_{\substack{x_i = \alpha s_i + (1-\alpha) t_i \\ \sum_{i\in T} s_i = z_1 \\ \sum_{i\in T} t_i = z_2}} \sum_{i\in T} v_i(x_i) \\
= & \max_{\substack{\sum_{i\in T} s_i = z_1 \\ \sum_{i\in T} t_i = z_2}} \sum_{i\in T} v_i(\alpha s_i + (1-\alpha) t_i) \\
\geq & \max_{\sum_{i\in T} s_i = z_1} \alpha \sum_{i\in T} v_i(s_i) + \max_{\sum_{i\in T} t_i = z_2} (1-\alpha) \sum_{i\in T} v_i(t_i) \\
= & \alpha \tilde{v}(z_1) + (1-\alpha)\tilde{v}(z_2)
\end{aligned}$$

where the inequality is due to Lemma 1. ■

LEMMA 3. *$\tilde{v}(\bar{x})$ is an increasing function of $\bar{x}$.*

Proof. From Lemma 2, we know that $\tilde{v}(\bar{x})$ is a concave function maximized over a convex set and has a unique maximum. Let the maximizing x_i's in the definition of $\bar{v}(\bar{x})$ be $x_i = g_i(\bar{x}), i \in T$. If $\delta > 0$, then

$$\begin{aligned}
\tilde{v}(\bar{x} + \delta) &= \max_{x \in Z_T(\bar{x}+\delta)} \sum_{i\in T} v_i(x_i) \\
&\geq \sum_{i\in T} v_i \left(g_i(\bar{x}) + \frac{\delta}{N_T} \right) \\
&> \sum_{i\in T} v_i(g_i(\bar{x})) \\
&= \tilde{v}(\bar{x})
\end{aligned}$$

where the first inequality is due to the max and the second inequality is because $v_i(\cdot)$ is an increasing function. ■

LEMMA 4. *$\tilde{v}(\bar{x})$ is differentiable with respect to $\bar{x}$.*

Proof. We have

$$\tilde{v}(\bar{x}) = \max_{x \in Z_T(\bar{x})} \sum_{i\in T} v_i(x_i) \ .$$

To find the optimal allocation, we introduce the Lagrangian

$$\mathcal{L} = \sum_{i\in T} v_i(x_i) + \lambda \left(\bar{x} - \sum_{i\in T} x_i \right) + \sum_{i\in T} \mu_i x_i$$

and take the partial derivative with respect to x_i;

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \Rightarrow \quad v_i'(x_i) = \lambda - \mu_i \ .$$

If the i^{th} agent does not receive a positive share at the optimal allocation, we have

$$x_i = 0 \quad \Rightarrow \quad \mu_i = \lambda - v_i'(0) \geq 0$$

which states that the Lagrange multiplier λ must exceed the marginal valuation of the agent at zero. If the agent receives a positive allocation, we have

$$x_i > 0 \quad \Rightarrow \quad \mu_i = 0 \quad \Rightarrow v_i'(x_i) = \lambda \quad \Rightarrow \quad x_i = v_i'^{-1}(\lambda)$$

which states that the marginal valuation at the allocation must be the same as the Lagrange multiplier λ. Thus, λ can be thought to represent the marginal valuation of the team, and determines which members of the team get positive allocations. Summing over the positive allocations we have

$$\bar{x} = \sum_{i\in T} v_i'^{-1}(\lambda) \ .$$

Let us define

$$w_i(\lambda) = \begin{cases} 1 & \lambda \in (0, v_i'(1)) \\ v_i'^{-1}(\lambda) & \lambda \in [v_i'(1), v_i'(0)] \\ 0 & \lambda \in (v_i'(0), \infty) \end{cases} \tag{7}$$

and

$$A(\lambda) = \{ i \in T : v_i'(0) > \lambda \}.$$

Then, we define

$$w(\lambda) = \sum_{i \in A(\lambda)} w_i(\lambda) \tag{8}$$

which is a strictly decreasing continuous function for $\lambda \in (\underline{\lambda}, \bar{\lambda})$ where

$$\underline{\lambda} = w^{-1}(1) \qquad \bar{\lambda} = w^{-1}(0) = \max_{i\in T} v_i'(0) \ .$$

A graphical representation of w can be seen in Figure 1. Thus, we can define the following inverse, which allows us to calculate the Lagrange multiplier λ as a function of total allocation:

$$\lambda(\bar{x}) := w^{-1}(\bar{x}) \quad \bar{x} \in (0,1).$$

$\lambda(\bar{x})$ is twice differentiable if $v_i(x_i)$ are strictly concave and

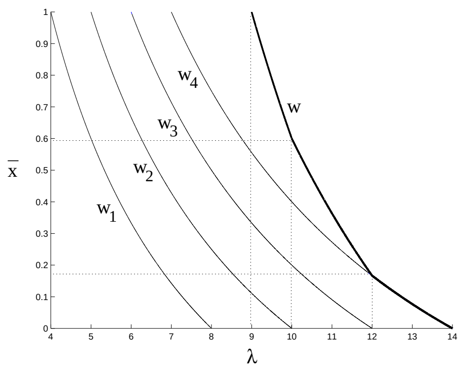

Figure 1: Graphical Representation of $w(\lambda)$

if $\bar{x} \neq w(v_i'(0)), \forall i \in T$. Assuming that the previous is true, we have

$$\begin{aligned}
\tilde{v}(\bar{x}) &= \sum_{i\in T} v_i(x_i^*) = \sum_{i \in A(\lambda(\bar{x}))} v_i \left(v_i'^{-1}(\lambda(\bar{x})) \right) \\
\tilde{v}'(\bar{x}) &= \sum_{i \in A(\lambda(\bar{x}))} v_i' \left(v_i'^{-1}(\lambda(\bar{x})) \right) v_i'^{-1\prime}(\lambda(\bar{x}))\, \lambda'(\bar{x}) \\
&= \lambda(\bar{x})\ \lambda'(\bar{x}) \sum_{i \in A(\lambda(\bar{x}))} v_i'^{-1\prime}(\lambda(\bar{x})) \\
&= \lambda(\bar{x})\ w^{-1\prime}(\bar{x})\ w'(\lambda(\bar{x})) \\
&= \lambda(\bar{x})
\end{aligned}$$

where the last equality is due to the orthogonality property of derivatives of inverses. If $\bar{x} = w(v_i'(0))$, we have

$$\begin{aligned} & \lim_{\epsilon \to 0} \frac{\tilde{v}(\bar{x}+\epsilon) - \tilde{v}(\bar{x})}{\epsilon} \\ = & \lim_{\epsilon \to 0} \frac{\tilde{v}(\bar{x}+\epsilon) - [\tilde{v}(\bar{x}+\epsilon) + \tilde{v}'(\bar{x}+\epsilon)\epsilon + \tilde{v}''(\zeta)\epsilon^2]}{\epsilon} \\ = & \lim_{\epsilon \to 0} \tilde{v}'(\bar{x}+\epsilon) = \lim_{\epsilon \to 0} \lambda(\bar{x}+\epsilon) = \lambda(\bar{x}) \end{aligned}$$

where $\zeta \in (\bar{x}, \bar{x}+\epsilon)$. The first equality is obtained by using a Taylor expansion of $\tilde{v}(\bar{x})$ around $\bar{x}+\epsilon$. This is done because we can make the substitution $\tilde{v}'(\bar{x}+\epsilon) = \lambda(\bar{x}+\epsilon)$ for all $\epsilon > 0$ where $\bar{x}+\epsilon \neq w(v_i'(0))$. ■

PROPOSITION 3. *$\tilde{v}(\bar{x})$ is a concave, increasing, differentiable function of $\bar{x}$.*

Proof. This is a result of Lemmas 1, 2, 3, and 4. ■

Thus, we have been able to show that a team of agents acting as one performs better for any given level of competition and the team can be represented as a single agent with a modified valuation function that satisfies Assumption 1.

4. EFFECTS OF COALITIONS

In this section, we identify the effects of joining a coalition on an agent's ability to participate in an auction from which it was previously excluded. We also address the effects of a coalition on the utility of agents outside the team. We first address the issue of agents not receiving a positive allocation due to the level of competition for a resource. Can such an agent obtain a positive allocation by joining a coalition?

PROPOSITION 4. *Any agent that does not receive a positive allocation cannot alter that allocation by joining a coalition.*

Proof. If p denotes the current price of a resource and $v(x)$ is the valuation function of an agent, the agent will be unable to participate if and only if $p \geq v'(0)$. Let us assume that the agent forms a coalition with other agents who also did not receive a positive allocation. Then, we have

$$\tilde{v}'(0) = \lambda(0) = w^{-1}(0) = \max_{i \in T} v_i'(0) \ .$$

But since $v_i'(0) \leq p$, $\forall i$, $\tilde{v}'(0) \leq p$ and the team will not be able to participate. Let us now assume that an agent joins a coalition that currently has a positive allocation. Let $\tilde{v}_1$ and $\tilde{v}_2$ denote the valuation functions of the team before and after the agent has joined. We know that at price p and allocation $\bar{x}_1$, a unique Nash equilibrium is formed with the remaining agents, and the utility function derived from the valuation $\tilde{v}_1$ is maximized. Let the Lagrange multiplier λ_1 be the value that determines the allocations to the members of the team under this scenario. Then, we have

$$\begin{aligned} \tilde{v}_1'(\bar{x}_1)(1-\bar{x}_1) = \lambda_1(1-\bar{x}_1) = p \geq v'(0) \\ \Rightarrow \quad \lambda_1 = \frac{v'(0)}{1-\bar{x}_1} > v'(0). \end{aligned}$$

From this we can tell that the Lagrange multiplier λ_1 also yields the maximizing allocations for $\tilde{v}_2$ for a total allocation of $\bar{x}_2 = \bar{x}_1$ as it satisfies all the necessary conditions of the optimization problem with the additional allocation of zero to the newly added agent since $\lambda_1 > v'(0)$. This implies that the price-allocation pair $(\lambda_1(1-\bar{x}_1), \bar{x}_1)$, which we know yields the unique Nash equilibrium response with respect to the remaining agents, will be the operating point for the team with valuation function $\tilde{v}_2$ and the added agent receives a zero allocation under this scenario as well. ■

The preceding Lemma states that the population of agents bidding for a resource is effectively separated into two classes: those who receive service and those who cannot under any circumstance given the current market conditions. Agents who do not receive allocation know immediately that they should either find an alternate resource with less competition or wait for the demand for the resource to decrease as no negotiation can help. Agents who receive service can limit their negotiations to only the agents currently receiving service.

For the agents who have an incentive to form coalitions, the value that they assign to a possible coalition depends on what they believe about the response of those outside the coalition. A classical approach would assign a value that is equal to the lower value of two-person zero-sum game between the coalition and those outside the coalition with the objective function being the sum of the utilities of those in the coalition. This gives a worst case estimate of the utility gained by the coalition. It assumes that those outside the coalition will view the coalition as a threat and will act in a manner to discourage its formation. In our auction, the value of a coalition would be

$$V_T = \max_{s_T} \min_{s_{-T}} \left\{ \tilde{v}\left(\frac{s_T}{s_T + s_{-T}} \right) - s_T \right\} .$$

If there is no penalty for not participating ($\tilde{v}(0) = 0$), we will have a value of zero for every coalition except the coalition of all the agents. This is because the opposition can let $s_{-T} \to \infty$, unless the bids are bounded. One could consider a value where the opponents of a team make the largest bid they can possibly make while keeping their utility nonnegative, which would bound the opponent's maximum bid. This again requires the assumption that the opponents want to discourage the formation of a coalition. We can show that, under the structure of our auction, there is no incentive to discourage coalitions.

LEMMA 5. *At a given price $\hat{p}$, the optimal demand of a team of agents is less than the sum of the optimal demands of agents acting individually.*

Proof. First, we consider the team of agents. The optimal demand at price $\hat{p}$ must satisfy the following function:

$$\hat{p} = \tilde{v}'(\bar{x})(1-\bar{x}) = \lambda(\bar{x})(1-\bar{x}) \ .$$

The set of $(\lambda, \bar{x})$ pairs that satisfy the previous equation can be represented by the following curve:

$$\bar{x} = 1 - \frac{\hat{p}}{\lambda} =: h(\lambda) \quad \lambda \in (\hat{p}, \infty) \ .$$

Let $w(\lambda)$ be defined as in (8). Then, w also characterizes the $(\lambda, \bar{x})$ pairs that maximize $\tilde{v}(\bar{x})$. Since h is an increasing function of λ and w is a decreasing function of λ, there is a unique value, say $\hat{\lambda}$, at which they are equal:

$$w(\hat{\lambda}) = 1 - \frac{\hat{p}}{\hat{\lambda}}.$$

The total demand for the team for the price $\hat{p}$ will be $\bar{x} = w(\hat{\lambda})$, with individual allocations $x_i = v_i'^{-1}(\hat{\lambda})$. We now consider the optimal demand when the agents maximize their individual utilities. We will denote this demand as y_i, to differentiate it from the allocation that an agent receives under team optimization. At the price $\hat{p}$, the optimal demand of an agent must satisfy

$$\hat{p} = v_i'(y_i)(1 - y_i).$$

Interpreting this as a team of one agent, $v_i(y_i)$ is optimized by $(\lambda, \bar{x})$ pairs on the curve $w_i(\lambda)$ as defined in (7). This must again intersect the curve h, which consists of all the $(\lambda, \bar{x})$ pairs yielding a price $\hat{p}$. At $\lambda = \hat{\lambda}$, we have

$$w_i(\hat{\lambda}) \leq w(\hat{\lambda}) = 1 - \frac{\hat{p}}{\hat{\lambda}} \ . \qquad (9)$$

Let $\check{\lambda}$ be the value where

$$w_i(\check{\lambda}) = v_i'^{-1}(\check{\lambda}) = 1 - \frac{\hat{p}}{\check{\lambda}} \ .$$

Since $w_i(\lambda)$ is a decreasing function of λ and h is an increasing function of λ, (9) implies that $\check{\lambda} < \hat{\lambda}$. Since $\check{\lambda}$ characterizes the optimal demand for the agent acting individually, and $v_i'^{-1}$ is a decreasing function, we have

$$y_i = v_i'^{-1}(\check{\lambda}) > v_i'^{-1}(\hat{\lambda}) = x_i \ .$$

Summing over all agents, we have

$$\bar{y} = \sum_{i \in T} y_i > \sum_{i \in T} x_i = \bar{x}$$

which states that the optimal total demand of agents acting individually is greater than the optimal total demand of agents acting as a team. A graphical representation of the proof can be seen in Figure 2. ■

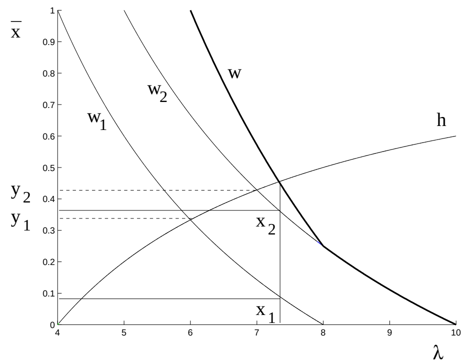

Figure 2: Agents in Teams Demand Less Than Agents Acting Individually

LEMMA 6. *Given the same set of opponents, the total bid of a team at equilibrium will be less than the total bids of the equilibrium reached when members of the team bid with respect to individual utilities.*

Proof. Let $d_I(p)$ be the sum of the demand functions of the agents acting individually, i.e., $d_I(p) = \sum_{i \in T} d_i(p)$ where $d_i(p)$ is the inverse of $p_i(x_i) = v_i'(x_i)(1 - x_i)$. Let $d_T(p)$ be the demand function associated with the team, i.e., $d_T(p)$ is the inverse of $p(\bar{x}) = \tilde{v}'(\bar{x})(1 - \bar{x})$. Let $d_{-T}(p)$ be the sum of all demand functions of all agents outside the team. From Lemma 5, we know

$$d_I(p) \geq d_T(p) \ \ \forall \ p \ .$$

The equilibrium price when agents act individually is given by

$$d_I(p_I) + d_{-T}(p_I) = 1 \ .$$

Since the demand function of the team is less than the sum of the individual demands, we have

$$d_T(p_I) + d_{-T}(p_I) \leq 1 \ .$$

If p_T is the equilibrium price of the team obtained from

$$d_T(p_T) + d_{-T}(p_T) = 1$$

then the fact that demand functions are decreasing implies that $p_T \leq p_I$. From this, we have

$$\begin{aligned} d_{-T}(p_T) &\geq d_{-T}(p_I) \\ 1 - d_{-T}(p_T) &\leq 1 - d_{-T}(p_I) \\ p_T(1 - d_{-T}(p_T)) &\leq p_I(1 - d_{-T}(p_I)) \ . \end{aligned}$$

The LHS of the previous equation is the equilibrium price for the team multiplied by the equilibrium demand of the team which is the equilibrium bid. Similarly, the RHS is the equilibrium bid of all the agents acting individually and we can see that the team bids less. A graphical representation of the proof can be seen in Figure 3. ■

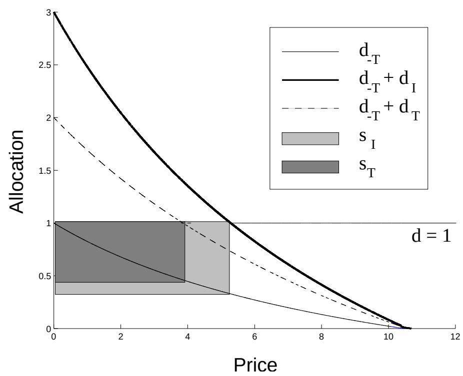

Figure 3: Agents in Teams Bid Less Than Agents Acting Individually

LEMMA 7. *At equilibrium, an agent's optimal utility increases as its opponents' bid total decreases at equilibrium.*

Proof. Let us define $f(t) = \arg\min_s U(s;t) = v(s/(s+t)) - s$, and let $t_1 \leq t_2$. Then,

$$\begin{aligned} U(f(t_2);t_2) &= v\left(\frac{f(t_2)}{f(t_2)+t_2}\right) - f(t_2) \\ &\leq v\left(\frac{f(t_2)}{f(t_2)+t_1}\right) - f(t_2) \\ &= U(f(t_2);t_1) \\ &\leq U(f(t_1);t_1) \end{aligned}$$

where the first inequality is because v is an increasing function of its argument and the second inequality is due to the definition of f. ■

PROPOSITION 5. *Given a set of agents at equilibrium, if a coalition is formed, then the utilities of all agents outside the coalition cannot decrease at the new equilibrium point brought about by the formation of the coalition.*

Proof. By Lemma 5, the formation of any team will reduce the total demand function faced by any agent outside the coalition. By Lemma 6, this will result in the opponents of that agent bidding less at the new Nash equilibrium. By Lemma 7, the agent will have a higher utility at the new equilibrium when facing a smaller sum of opponents' bids. ■

The previous proposition states that when any set of agents form a coalition, all other agents benefit because the coalition reduces the price for everyone. This makes the notion of a punitive response to a coalition invalid. Thus, an appropriate value of a coalition would be the utility of the team at the unique Nash equilibrium when opponents act solely to maximize their own utility.

5. CONCLUSION

We have investigated coalition formation in a proportionally fair divisible auction. When agents form teams, we have shown that they can be represented as a single agent with a modified valuation function. If an agent is excluded from service in the purely noncooperative game, it cannot obtain service by joining or forming any coalition. This separates the agents into two classes. This simplifies the decisions facing excluded agents and reduces the contact set (agents with which one might want to make a coalition) for those receiving service. Furthermore, we have shown that the formation of a coalition increases the performance of those outside the team, creating an environment promoting cooperation. Since the suggested value of a coalition would be taken when opponents act in self-interest, calculating the value would be more difficult as a team would need to have information about opponents' valuation functions to obtain the value of the team. This also raises interesting questions about whether it is more beneficial to form a coalition or to wait for others to form a coalition. The order of how teams are formed may be important. Benefits might also depend on how the excess utility gained by forming a team would be divided. Another issue is the tradeoff between the revenue generated for the resource and the social welfare of the agents as every coalition decreases the revenue generated while increasing the sum of agents' utilities.

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