Equilibrium and Negotiation in Multiple Resource Auctions

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***Abstract*—**
The application of economic ideas to control electronic technology is an emerging design paradigm. In network and computational settings, divisible auctions are a viable tool for regulation of services such as bandwidth and processing share. Often, the performance of an agent in the system depends simultaneously on the allocation received from several resources. In this paper, we investigate the extension of a proportionally fair divisible auction to the case where agents' utilities have complementarities across multiple resources. We show the existence and uniqueness of a Nash equilibrium under various symmetry conditions. We propose a negotiation algorithm and prove that under relaxation, the scheme is locally stable for particular agent characterizations.

I. Introduction

Market mechanisms have been advocated as an effective method to control electronic resources for many of the same reasons that markets have effectively regulated the trade of traditional goods [4], [6]. Currency-based models allow for common valuation of heterogeneous resources giving system managers or agents the ability to establish priority or specify preferences. Markets are appropriate for decentralized systems because independent allocations can occur simultaneously in a distributed network without a central authority. In addition, prices serve as low-dimensional feedback for control. Examples of market-based control can be found in factory scheduling, manufacturing systems, energy distribution and pollution management [2].

We focus on auctions over other pricing mechanisms because the transparency of the allocation rule removes any information advantage of knowing how a price rule is derived. Most classic auction literature focuses on the sale of a single indivisible good [5]. Recently, there has been a lot of work on the auction of multiple units of indivisible goods [3]. However, neither of these models are a good fit for electronic technology. Resources such as bandwidth and processing share, though not infinitely divisible, are offered in large quantities (Gb/s, MHz) that can be arbitrarily partitioned. To address this, we have studied a proportionally divisible auction of a single resource [8]–[11]. Proportional allocation has been advocated for scheduling due to its ease of implementation and many such systems have been designed [7], [12], [13].

The performance of agent tasks in electronic settings often depend on access to more than one resource. The Internet is a combination of several independent networks and creating a connection via bandwidth auctions might require obtaining service from different providers. In a computational setting, the performance of processes can depend on the obtained allocation of both CPU share and memory.

In this paper, we extend the proportionally fair divisible auction to a multiple resource setting. One of our goals is to study the nature of equilibria (existence and uniqueness) under this structure. Characterizations of valuations for divisible auctions require large information loads so single shot resolutions have undesirable communication costs. Thus, we also investigate negotiation algorithms that converge to stable market clearing.

The paper is organized as follows. In Section II, we present the proportionally fair auction and describe agents' utilities that include complementarity across multiple resources. In Section III, we investigate the existence and uniqueness of equilibria. In Section IV, we propose a negotiation algorithm based on price feedback and analyze its convergence properties. In Section V, we discuss the notion of relaxability and show that a relaxed version of the proposed iteration scheme is locally stable for agent populations with various symmetries in their valuations. In Section VI, we summarize our work and present a few areas for future study.

II. System Model

We assume that there are K resources of interest to a particular population of N agents. Let s_i^k be a nonnegative real number which denotes the i-th agent's bid for the k-th resource. For now, we assume that every agent is bidding for every resource though this can be generalized to the case where each agent is interested in only a subset of the K resources. Let $s_{-i}^k := \sum_j s_j^k - s_i^k$. Then, s_{-i}^k is a measure of outside competition at resource k with respect to the i-th agent. We assume that each resource uses a proportionally fair divisible auction [8]–[11] for its allocation rule. Thus the i-th agent receives a fraction

$$x_i^k = \frac{s_i^k}{s_i^k + s_{-i}^k + \epsilon^k} \tag{1}$$

of the k-th resource. The cost to the i-th agent is

$$c_i = \sum_k s_i^k \tag{2}$$

which is the sum of all bids made by the i-th agent. In the allocation rule, $\epsilon^k > 0$ denotes the bid made by the resource itself. This serves both mathematical and practical purposes. Mathematically, $\epsilon^k > 0$ prevents the denominator from ever being zero in the allocation rule. Allowing a zero denominator becomes problematic if there is a single user who wants to gain access to the resource. Practically, the resource's bid prevents collusive behavior where at equilibrium the agents agree to scale down their bids by the same factor, thus maintaining the same allocation but at a lower cost. By bidding ϵ^k, the k-th resource effectively sets a minimum price for obtaining various levels of allocation. This is analogous to setting a reservation price in an indivisible auction context.

The agents are characterized by their valuation functions. If $v_i^k(x_i^k)$ denotes the benefit to the i-th agent from the k-th resource if it were to receive an allocation x_i^k, then the i-th agent's utility is

$$U_i = \min_k \left\{ v_i^k \left(\frac{s_i^k}{s_i^k + s_{-i}^k + \epsilon^k} \right) \right\} - \sum_k s_i^k. \tag{3}$$

This model of bottleneck utility can be applied in various electronic resource settings. In a network bandwidth auction, if an agent is trying to put together a connection of K links in series, where each link is auctioned separately, the value of the connection to the broker agent will be the minimum value of the bandwidth obtained among the K links. In a computational setting, the performance index could be the time to computation which is limited simultaneously by the allocation of processing, memory, access to a database or other resources relevant to the task at hand. The agents' valuations are assumed to be strictly increasing, concave functions to represent the principle of diminishing returns. In addition, an agent is not penalized for not obtaining access to the resource. Thus we have

$$v_i^k(\cdot) > 0, v_i^k(0) = 0, {v_i^k}'(\cdot) > 0, {v_i^k}''(\cdot) \le 0.$$

In this allocation mechanism, the i-th agent submits a bid vector $s_i = [s_i^1 \cdots s_i^K]$. Let λ_i be the benefit achieved by the i-th agent defined as follows:

$$\lambda_i = \min_k \left\{ v_i^k \left(\frac{s_i^k}{s_i^k + s_{-i}^k + \epsilon^k} \right) \right\}.$$

Let us assume that for some resource $\hat{k}$,

$$v_i^{\hat{k}} \left(\frac{s_i^{\hat{k}}}{s_i^{\hat{k}} + s_{-i}^{\hat{k}} + \epsilon^{\hat{k}}} \right) > \lambda_i.$$

Then, the i-th agent could reduce $s_i^{\hat{k}}$ while not affecting the achieved benefit yet reducing the cost yielding a higher utility. Thus, an optimal bid would be one such that the allocation would yield the same individual benefits from all the resources:

$$v_i^k \left(\frac{s_i^k}{s_i^k + s_{-i}^k + \epsilon^k} \right) = \lambda_i \quad \forall k.$$

A benefit of λ_i would be achieved by the bids:

$$s_i^k = \frac{z_i^k(\lambda_i)}{1 - z_i^k(\lambda_i)} (s_{-i}^k + \epsilon^k)$$

where $z_i^k(\lambda_i) = {v_i^k}^{-1}(\lambda_i)$ is the inverse of the valuation function. Thus, we have

$$0 \le z_i^k(\cdot) < 1, z_i^k(0) = 0, {z_i^k}'(\cdot) > 0, {z_i^k}''(\cdot) \ge 0.$$

The functions $\{z_i^k(\cdot)\}$ represent the allocation level necessary at a particular resource to achieve a given benefit. We can now express an agent's utility as a function of its benefit as follows:

$$U_i(\lambda_i) = \lambda_i - \sum_k \frac{z_i^k(\lambda_i)}{1 - z_i^k(\lambda_i)} (s_{-i}^k + \epsilon^k).$$

We can show that the second derivative of the utility U_i with respect to benefit λ_i is well defined and negative. This implies that the marginal utility will decrease as the benefit increases. The maximum utility is achieved when the marginal utility is zero. If $U_i'(0) \le 0$, then $\lambda_i = 0$ is the optimal benefit and the agent responds by bidding zero for all the resources. Thus, a necessary condition for positive bidding is:

$$1 - \sum_k {z_i^k}'(0)(s_{-i}^k + \epsilon^k) > 0 \quad \Rightarrow \quad \sum_k \frac{s_{-i}^k + \epsilon^k}{{v_i^k}'(0)} < 1.$$

If the previous condition is not met, it implies that the demand for the resources is too high for the agent to participate. The agent must either wait for the demand to subside or find alternate resources. For the previous condition to be satisfied, it is necessary that $\sum_k \frac{\epsilon^k}{{v_i^k}'(0)} < 1$. If the previous condition is not met, it implies that the resources are inherently too expensive for the agent. The sellers have valued their services at a level such that the agent cannot have positive utility under any bid vector. The agent's only choice is to find alternate resources.

We note that the optimality conditions depend on s_{-i}^k. Thus, our search for an equilibrium still occurs in bid-space which is of cardinality NK. It would be desirable to be able to characterize equilibrium in benefit space as it is only of cardinality N. To this end, let $p^k := \sum_i s_i^k + \epsilon^k$ be the sum of all bids. Given a set of bids and an allocation, the cost per unit resource at resource k is p^k for all agents. Therefore, we call p^k the *price* of the k-th resource. From this interpretation, we have

$$p^k = \frac{s_i^k}{z_i^k} = \frac{s_{-i}^k + \epsilon^k}{1 - z_i^k}.$$

We can now rewrite the marginal utility as follows:

$$U_i'(\lambda_i) = 1 - \sum_k \frac{z_i^{k\prime}(\lambda_i)}{1 - z_i^k(\lambda_i)} p^k.$$

From our allocation rule we have

$$\sum_i z_i^k(\lambda_i) + \frac{\epsilon^k}{p^k} = 1 \Rightarrow p^k = \frac{\epsilon^k}{1 - \sum_i z_i^k(\lambda_i)}.$$

Substituting for p^k in the marginal utility, we have

$$U_i'(\lambda_i) = 1 - \sum_k \frac{z_i^{k\prime}(\lambda_i)}{1 - z_i^k(\lambda_i)} \cdot \frac{\epsilon^k}{1 - \sum_i z_i^k(\lambda_i)}. \qquad (4)$$

We can now define an equilibrium for this auction game as a vector of benefits $[\lambda_1 \cdots \lambda_N]$ such that $U_i'(\lambda_i) = 0$ if $\lambda_i > 0$ and $U_i'(0) \leq 0$ if $\lambda_i = 0$. We will refer to the multiple resource auction game where the allocations are characterized by (1), the costs are characterized by (2) and agents' utilities are characterized by (3) as the MRPF (Multiple Resource Proportionally Fair) auction.

III. Existence and Uniqueness of Equilibrium

We now investigate whether an equilibrium solution exists and if so, the conditions under which that equilibrium is unique. Because the allocation by a particular agent is directly affected by the bids of all the other agents, the utilities of all agents are coupled through each others' actions. This nature of interaction invites game-theoretic analysis. Our equilibrium concept is the Nash equilibrium [1]. We seek a vector of bids (or benefits) from which no agent will choose to unilaterally deviate given that all other agents' bids (benefits) remain fixed. We can use a slight variation on Theorem 4.3 in Başar and Olsder [1] to show that the MRPF auction game yields a Nash equilibrium in pure strategies.

Once existence of equilibria has been established, we investigate whether a unique equilibrium exists. A unique equilibrium is desirable because it reduces uncertainty and simplifies the agents' decision process. If the equilibrium is reached, an agent need not question whether an alternate market clearing would have yielded a higher utility. Furthermore, if the equilibrium was reached through a dynamic process, there is no incentive to drive the negotiation to a particular state. This reduces complexity of strategies that an agent needs to consider and this is generally considered to be a beneficial trait of an auction. It has been speculated that the relative simplicity of the ascending (English) auction as compared to the theoretically equivalent second-price sealed bid auction accounts for the significant preference of the former over the latter. Uniqueness for a general case, which simulations seem to verify, has not yet been proven analytically. Thus, we investigate agent populations with various symmetries.

A. Symmetric Agents with Asymmetric Valuations

We now consider the case where all agents have symmetric valuation functions but the valuations across resources may be asymmetric, i.e.

$$z_i^k(\lambda_i) = z^k(\lambda_i) \; \forall k$$

This representation of a common value model could occur in a computational setting such as users sharing a local network of computers. If their tasks are similar, the value of processing and memory are identical across user agents even though each resource effects performance in different ways.

Proposition 1: In the MRPF auction game, if the agents have symmetric valuations which may be asymmetric across resources as denoted in (III-A), there is a unique equilibrium.
Proof. We modify the equilibrium condition for the i-th user to obtain:

$$\sum_k \frac{z^{k\prime}(\lambda_i)}{1 - z^k(\lambda_i)} \cdot \frac{\epsilon^k}{1 - \sum_j z^k(\lambda_j)} = 1.$$

For a given candidate equilibrium solution we have a benefit vector λ. Letting,

$$p^k(\lambda) := \frac{\epsilon^k}{1 - \sum_j z^k(\lambda_j)}$$

we have

$$\sum_k \frac{z^{k\prime}(\lambda_i)}{1 - z^k(\lambda_i)} \cdot p^k(\lambda) = 1$$

as the equilibrium condition for positive benefits. Given a candidate vector λ, if $\lambda_i > \lambda_j$, we have

$$\frac{z^{k\prime}(\lambda_i)}{1 - z^k(\lambda_i)} > \frac{z^{k\prime}(\lambda_j)}{1 - z^k(\lambda_j)} \; \forall k$$

which implies

$$1 = \sum_k \frac{z^{k\prime}(\lambda_i)}{1 - z^k(\lambda_i)} p^k(\lambda) > \sum_k \frac{z^{k\prime}(\lambda_j)}{1 - z^k(\lambda_j)} p^k(\lambda) = 1$$

which is a contradiction. As in the totally symmetric case, any equilibrium benefit vector must be of the form $\lambda = a1_N$. Considering $\lambda = a1_N, \hat{\lambda} = b1_N, a > b$, we have $p^k(a) > p^k(b) \; \forall k$, which along with the properties of $z^k(\cdot)$ imply that

$$1 = \sum_k \frac{z^{k\prime}(a)}{1 - z^k(a)} \cdot p^k(a) > \sum_k \frac{z^{k\prime}(b)}{1 - z^k(a)} \cdot p^k(b) = 1$$

which is a contradiction. Again, we must consider the possibility that at equilibrium a strict subset of the agents may have zero benefit. Let I denote the subset of agents with zero benefit. Then, the equilibrium condition for agents in I yields

$$\sum_k z^{k\prime}(0) p^k \geq 1.$$

From the equilibrium condition for agents in I^C, we have

$$1 = \sum_k \frac{{z^k}'(\lambda_i)}{1 - z^k(\lambda_i)} p^k > \sum_k {z^k}'(\lambda_i) p^k \geq \sum_k {z^k}'(0) p^k \geq 1$$

which is a contradiction. For this case, we also have a unique equilibrium solution. ■

B. Asymmetric Agents with Symmetric Valuations

In the previous cases, the equilibrium solutions were composed of symmetric benefits and hence, symmetric bids across agents. To see if uniqueness holds in cases where the equilibrium solutions are not necessarily symmetric across agents we investigate the case where

$$z_i^k(\lambda_i) = z_i(\lambda_i) \ \forall k.$$

This occurs when there is a certain qualitative homogeneity among the resources being allocated. An example of this would be a network bandwidth auction where K links in series are being allocated separately and N broker agents are attempting to obtain a connection across the links. Because all the resources in contention are bandwidth and the final valuation is based only on the size of the connection, the agent's performance is determined by the minimum level of allocation across resources, i.e.

$$\lambda_i = v_i \left(\min_k \{x_i^1, \cdots, x_i^K\} \right).$$

Each agent has a different valuation for the allocation due to heterogeneity in the application being catered to or the wealth of the consumer base to whom the broker agents are selling the connection. However, the valuations are identical across the resources.

Proposition 2: In the MRPF auction game, if the agents have asymmetric valuations which are symmetric across resources as denoted in (III-B), there is a unique equilibrium.

Proof. The equilibrium condition for the i-th agent is

$$\frac{{z_i}'(\lambda_i)}{1 - z_i(\lambda_i)} \cdot \frac{\sum_k \epsilon^k}{1 - \sum_j z_j(\lambda_j)} = 1.$$

Given $y \in (z_i'(0), \infty)$,

$$\frac{{z_i}'(\lambda_i)}{1 - z_i(\lambda_i)} = y$$

has a unique solution in λ_i because the LHS of the previous equation is strictly increasing in λ_i and y is in its range space. Obtaining solutions for all $y \in (z_i'(0), \infty)$, we obtain the function $\lambda_i(y)$ which is a strictly increasing function of y. Let $\lambda_i(y) = 0$ for $y \in [0, z_i'(0)]$. Then, we have that $1 - \sum_j z_j(\lambda_j(y))$ is strictly decreasing in y on $(\min_i z_i'(0), \infty)$ and equal to one on $[0, \min_i z_i'(0)]$. Then,

$$1 - \sum_j z_j(\lambda_j(y)) = y \sum_k \epsilon^k$$

has a unique solution $y^* \in (0, \infty)$ as the RHS of the previous equation is strictly increasing and goes to infinity and $1 - \sum_j z_j(\lambda_j(0)) = 1 > 0$. If $y^* \leq z_i'(0)$, then

$$z_i'(0) \frac{\sum_k \epsilon^k}{1 - \sum_j z_j(\lambda_j)} \geq y^* \frac{\sum_k \epsilon^k}{1 - \sum_j z_j(\lambda_j)} = 1 \Rightarrow \lambda_i = 0.$$

If the agents are ordered such that

$$z_1'(0) < z_2'(0) < \cdots < z_N'(0),$$

then, we have a separation between active and inactive agents as follows:

$$\begin{array}{lcl} y^* \in [0, z_1'(0)] & \Rightarrow & \text{No active agents} \\ y^* \in (z_1'(0), z_2'(0)] & \Rightarrow & \{1\} \text{ is active} \\ y^* \in (z_2'(0), z_3'(0)] & \Rightarrow & \{1, 2\} \text{ are active} \\ y^* \in (z_{N-1}'(0), z_N'(0)] & \Rightarrow & \{1, \ldots, N-1\} \text{ are active} \\ y^* \in (z_N'(0), \infty) & \Rightarrow & \text{All agents are active} \end{array}$$

If the i-th agent is active, then $\lambda_i^* = \lambda_i(y^*)$. Thus, given any set of valuation functions, we obtain a unique y^*, and from y^* we obtain a unique benefit vector λ which implies that we have a unique equilibrium solution. ■

IV. Fixed Price Iteration

Given that an equilibrium exists for a market mechanism, an immediate question is how one arrives at that state. If we were operating in a centralized regime where the large signaling costs, large computational loads and wholesale preference revelation were not issues, agents could submit curves that characterize their valuations or optimal responses to a central authority (the seller, for example) and have that authority calculate and enforce the equilibrium point. However, we apply an auction mechanism with the idea that allocations (and consequently equilibrium) are the responsibility of the agent and thus, we investigate iterative negotiation schemes based on feedback.

In our model, a natural quantity to be fed back is the price of each resource p^k. We consider a scheme where agents alter their bids assuming that the price of the resource will remain the same at the next stage. At equilibrium, we have

$$\sum_k \frac{{z_i^k}'(\lambda_i^*)}{1 - z_i^k(\lambda_i^*)} {p^k}^* = 1, \ \forall i.$$

Given price feedback $p^{(n)} = [{p^1}^{(n)} \cdots {p^K}^{(n)}]$ at stage n, the i-th agent calculates a benefit $\lambda_i^{(n)}(p^{(n)})$ from

$$\sum_k \frac{{z_i^k}'(\lambda_i^{(n)})}{1 - z_i^k(\lambda_i^{(n)})} {p^k}^{(n)} = 1. \qquad (5)$$

This yields a unique solution $\lambda_i^{(n)} := \lambda_i^{(n)}(p^{(n)})$ as the LHS of (5) is a strictly increasing function of $\lambda_i^{(n)}$. If

$\sum_k {z_i^k}'(0)p^{k(n)} \geq 1$, then $\lambda_i^{(n)} = 0$. Bids at stage $n+1$ are obtained from

$$s_i^{k(n+1)} = z_i^k(\lambda_i^{(n)})p^{k(n)}.$$

Summing over agents, we have

$$p^{k(n+1)} = \sum_i z_i^k(\lambda_i^{(n)})p^{k(n)}$$

which characterizes the evolution of prices. Taking partial derivatives with respect to $p^{l(n)}$, we have

$$\frac{\partial p^{k(n+1)}}{\partial p^{l(n)}} = \sum_i {z_i^k}'(\lambda_i^{(n)})\frac{\partial \lambda_i^{(n)}}{\partial p^{l(n)}}p^{k(n)} + \delta_{kl}\sum_i z_i^k(\lambda_i^{(n)}).$$

Treating (5) as an identity with $\lambda_i^{(n)}$ as a function of $p^{l(n)}$, and taking partial derivatives w.r.t. $p^{l(n)}$, we can obtain. $\frac{\partial \lambda_i^{(n)}}{\partial p^{l(n)}}$. Consequently, we can represent the price iteration Jacobian as follows:

$$J = -P\sum_i y_i y_i^T \frac{A_i}{c_i} + D$$

$$\text{where } c_i = \sum_{k=1}^{K} p^{k*}\left[\frac{{z_i^k}''(\lambda_i^*)}{1-z_i^k(\lambda_i^*)} + \left(\frac{{z_i^k}'(\lambda_i^*)}{1-z_i^k(\lambda_i^*)}\right)^2\right],$$

$$y_i = \begin{bmatrix} {z_i^1}'(\lambda_i^*) \\ \vdots \\ {z_i^K}'(\lambda_i^*) \end{bmatrix}, A_i = \text{Diag}\left(\frac{1}{1-z_i^j(\lambda_i^*)}\right),$$

$$P = \text{Diag}\left(p^{j*}\right),\ D = \text{Diag}\left(\sum_i z_i^j(\lambda_i^*)\right)$$

where Diag $(d(j))$ denotes a matrix where only the diagonal terms are nonzero and the j-th diagonal term is $d(j)$. If the eigenvalues of J are inside the unit disk, the fixed price iteration scheme converges locally to a Nash equilibrium.

V. Relaxability

Simulations of the fixed price iteration scheme show that it does not always converge to a stable equilibrium. This occurs when agents repeatedly "overreact" to market conditions. This phenomenon is also found in the single resource proportionally fair auction [8], [9], [11]. One possible solution is *relaxation* where an agent uses the following iteration scheme

$$s_i^{k(n+1)} = \alpha f_i^k(s^{(n)}) + (1-\alpha)s_i^{k(n)}$$

where $\alpha \in (0,1)$ and $f_i^k(s^{(n)})$ is the projected bid from the unrelaxed scheme. We call an iteration scheme *relaxable* if there exists an $\alpha \in (0,1)$ such that the eigenvalues of the relaxed Jacobian $\tilde{J} = \alpha J + (1-\alpha)I$ are inside the unit disk. How do we know if a Jacobian J is relaxable?

Proposition 3: For an iteration scheme to be relaxable, it is necessary and sufficient that all eigenvalues of the unrelaxed Jacobian J have real parts less than one.

Outline of Proof. The basic idea is that as a system is more relaxed (i.e. $\alpha \to 0$), each eigenvalue moves closer to the point $(1,0)$ on the complex plane along the trajectory created by the line segment between $(1,0)$ and the value of the original eigenvalue. If the real part of the original eigenvalue is greater than or equal to one, the line segment will remain outside the unit disk. Furthermore, if the real part of the original eigenvalue is less than one, then there exists some α_0 for each original eigenvalue such that $\forall \alpha < \alpha_0$, the corresponding eigenvalue of the relaxed system will be inside the unit disk. Thus, we can choose α sufficiently small such that all eigenvalues of the relaxed system lie inside the unit disk. ■

We now investigate whether the fixed point iteration scheme is relaxable when applied to the MRPF auction game. We consider the symmetric agent/ asymmetric resource valuation case described in Section III-A and the asymmetric agent / symmetric resource valuation case described in Section III-B.

Proposition 4: The fixed point iteration scheme is relaxable for the MRPF auction, when agents have symmetric valuations which are not necessarily symmetric across resources.

Proof. We have $z_i^k = z_j^k, \forall i,j$ which implies that

$$A_i = A_j = A, \quad c_i = c_j = c, \quad y_i = y_j = y \ \ \forall i,j$$

The Jacobian can then be expressed as

$$J = -\frac{1}{c}P\sum_i yy^T A + D = -\left(\frac{N}{c}\right)Pyy^T A + D.$$

The matrix J is symmetrizable ($J^T M = MJ$) with the matrix $M = AP^{-1}$. Hence, the eigenvalues of J are real. Letting $\gamma = -N/c$, $\hat{\lambda}$ be a candidate eigenvalue and x be the corresponding eigenvector. We have

$$J = -\gamma Pyy^T A + D \ \Rightarrow\ Jx = -\gamma Py(y^T Ax) + Dx = \hat{\lambda}x.$$

We note that all components of $y^T A$ are positive. If d_k is the k-th diagonal component of D and x_k is the k-th component of x, we have

$$-\gamma p^{k*}{z^k}'(y^T Ax) = (\hat{\lambda} - d_k)x_k.$$

If $y^T Ax \geq 0$, then $\hat{\lambda} > \max_k d_k \ \Rightarrow\ x_k \leq 0\ \forall k \ \Rightarrow$ $y^t Ax < 0$, which is a contradiction. If $y^T Ax \leq 0$, then $\hat{\lambda} > \max_k d_k \ \Rightarrow\ x_k \geq 0\ \forall k \ \Rightarrow\ y^t Ax > 0$ which is a contradiction. Thus, any viable $\hat{\lambda}$ must satisfy $\hat{\lambda} < \max_k d_k = \max_k \sum_i z_i^k < 1$. Applying Proposition 3, we have that the iteration scheme is relaxable. ■.

Proposition 5: The fixed point iteration scheme is relaxable for the MRPF auction, when agents have asymmetric valuations which are symmetric across resources.

Proof. We have $z_i^k = z_i^l, \forall k,l$ which implies

$$y_i = z_i' 1_K, \quad A_i = \frac{1}{1-z_i}I.$$

The Jacobian can then be expressed as

$$\begin{aligned} J &= -P\sum_i z_i' 1_K 1_K^T \frac{z_i'}{1-z_i}\frac{1}{c_i} I + D \\ &= -P\sum_i \frac{(z_i')^2}{c_i(1-z_i)} 1_K 1_K^T + D \\ &= -\gamma P 1_K 1_K^T + D \end{aligned}$$

where $\gamma = \sum_i \frac{(z_i')^2}{c_i(1-z_i)} > 0$. By symmetry, $s_i^{k^*} = s_i^{l^*}\ \forall k,l \Rightarrow p^{k^*} = p^{l^*}\ \forall k,l$. Thus, J is a symmetric matrix and its eigenvalues are real. Even if the prices were not identical, the matrix is symmetrizable with $M = P^{-1}$. If $\hat{\lambda}$ is a candidate eigenvalue and x is the corresponding eigenvalue, we have

$$-\gamma p^{k^*}\bar{x} + d_k x_k = \hat{\lambda} x_k$$

where $\bar{x} = \sum_k x_k$ and d_k is the k-th diagonal term of D. This can be rewritten as

$$-\gamma p^{k^*}\bar{x} = (\hat{\lambda} - d_k) x_k.$$

If $\bar{x} \geq 0$, then $\hat{\lambda} > \max_k d_k \ \Rightarrow\ x_k \leq 0\ \forall k \ \Rightarrow\ \bar{x} < 0$. If $\bar{x} \leq 0$, then $\hat{\lambda} > \max_k d_k \ \Rightarrow\ x_k \geq 0\ \forall k \ \Rightarrow\ \bar{x} > 0$. Thus, any viable $\hat{\lambda}$ must satisfy $\hat{\lambda} < \max_k d_k = \max_k \sum_i z_i^k < 1$. Applying Proposition 3, we have that the iteration scheme is relaxable. ■

VI. Summary and Future Work

In this paper, we have analyzed a multiple resource extension of a proportionally fair divisible auction. Though agents' utilities have complementarities across allocations from independently auctioned resources, we have shown that equilibria exist. Furthermore, a unique equilibrium have been proven to exist under various conditions of symmetry in the valuations of the agent population. In divisible auctions, resolutions after a single exchange of information cannot be achieved without large overhead. Thus, we introduced a negotiation algorithm that converges locally when sufficiently relaxed. Simulations have shown that uniqueness of the Nash equilibrium and local convergence of the relaxed iterative scheme hold for general agent populations not discussed in the paper. Analytical verification of this is an area of future research as is the investigation of alternate forms of utility complementarity.

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